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Compactifications of conformal gravity

Ignacio Navarro and Karel Van Acoleyen

IPPP, University of Durham
DH1 3LE Durham, U.K.
E-mail: ignacio.navarro@durham.ac.uk, karel.van-acoleyen@durham.ac.uk

Abstract: We study conformal theories of gravity, *i.e.* those whose action is invariant under the local transformation $g_{\mu\nu} \to \omega^2(x) g_{\mu\nu}$. As is well known, in order to obtain Einstein gravity in 4D it is necessary to introduce a scalar compensator with a VEV that spontaneously breaks the conformal invariance and generates the Planck mass. We show that the compactification of extra dimensions in a higher dimensional conformal theory of gravity also yields Einstein gravity in lower dimensions, without the need to introduce the scalar compensator. It is the field associated with the size of the extra dimensions (the radion) that takes the role of the scalar compensator in 4D. The radion has in this case no physical excitations since they are gauged away in the Einstein frame for the metric. In these models the stabilization of the size of the extra dimensions is therefore automatic.

Keywords: Field Theories in Higher Dimensions, Conformal and W Symmetry, Classical Theories of Gravity, Models of Quantum Gravity.

Contents

1. Introduction: conformal invariance in 4D

Symmetries play a central role in physics and amongst them local symmetries are especially important: they reflect the redundancies that the introduction of coordinates in spacetime or field space inevitably produce. Local symmetries form the skeleton of our description of particle interactions since a quantum field theory involving spin one fields needs gauge symmetry for consistency. In the same fashion, invariance of the action under diffeomorphisms is the basis of our understanding of gravity.

However symmetries are often spontaneously broken, or hidden, in our universe. This happens when the vacuum (or some "parameters" of our low energy lagrangian) transform under the symmetry in question. This phenomenon occurs with the electroweak interactions of the standard model, where the gauge group $\mathrm{SU}(2)\times\mathrm{U}(1)$ is spontaneously broken down to the electromagnetic $\mathrm{U}(1)$, and how and why it takes place are probably the most prominent questions in elementary particle theory. In this sense one could also say that diffeomorphism invariance is spontaneously broken down to Poincaré invariance in Minkowski spacetime, since one can see the metric as a field with a vacuum expectation value (VEV), breaking the diffeomorphism invariance down to the isometry group of spacetime.

An interesting question is then if there are other non-apparent local symmetries under which the spacetime metric transforms. In this letter we consider one such possibility: conformal theories of gravity, *i.e.* those whose action is left invariant under the transformation

$$g_{\mu\nu} \to \omega(x)^2 g_{\mu\nu} \tag{1.1}$$

for any smooth non-zero $\omega(x)$. Notice that this is not a coordinate transformation: although some coordinate transformations (dilatations) can have a similar effect on the metric, this transformation is not related to a change of coordinates, it is a genuine new local internal symmetry of the metric.

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We will only study classical solutions of conformally invariant theories. At the quantum level the conformal symmetry, like gauge symmetries, is anomalous. So if one wants to embed this symmetry into a realistic model of particle physics, one would have to show that this anomaly can be cancelled, as happens with the gauge symmetries of the Standard Model (SM). The issues related to the quantization of (conformal) gravity are however beyond the scope of this letter, but let us just mention that conformal invariance is regarded as a property of the theory making the gravitational quantum corrections more tractable and has been linked to the existence of an ultraviolet renormalization group fixed point [1]. Remember that since the metric produces our local units of measure, this symmetry seems to be in conflict with any dimensionful parameter of the lagrangian, so dimensionful parameters should conceal fields that transform under this rescaling.

It is time to recall that we could not write down a realistic lagrangian for describing the world without at least two dimensionful parameters: the scale of electroweak symmetry breaking ($M_{ew} \sim 10^2\, GeV$), given by the Higgs mass in the SM and the Planck mass ($M_p \sim 10^{18} GeV$), controlling the strength of gravity. To these we could add the vacuum energy scale, apparently of the order of $\Lambda_{vac} \sim (10^{-3} eV)^4$ [2]. The hugely different magnitudes of these scales have been disturbing theoretical physicists for years, and no fully convincing explanation for these enormous hierarchies has been put forward so far. The dimensionful parameters of the lagrangian describing the universe are thus the most mysterious ones, especially if we consider the quantum corrections to the theory that naively would seem to contribute to all of them with large and similar amounts. In this context, the conformal symmetry could have interesting implications,[1] since it links all mass scales to a common origin, the breaking of conformal invariance.[2] Notice also the relation $M_{ew}^2 \sim M_p \Lambda^{1/4}$ that could be interpreted as pointing to a common origin of these mass scales.

If we want to build a conformally invariant lagrangian for gravity in 4D that involves only the metric, the only option we have is Weyl gravity

$$S_{\text{gravity}} \propto \int \sqrt{g} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} d^4x \tag{1.2}$$

where $C^{\mu}{}_{\nu\lambda\rho}$ is the Weyl tensor, that is invariant under the transformation (1.1). This theory was proposed by Weyl [5] back in the early days of general relativity. In a series of papers Mannheim and Kazanas [6], have put forward Weyl gravity as an alternative to Einstein gravity that could also explain the galaxy rotation curves without the need for dark matter, but such a theory leaves many unanswered questions and it is not clear that it can provide a realistic alternative to General Relativity [7]. Furthermore, even if we give the Higgs field and the fermion fields weights under the conformal transformation:

$$h \to \omega(x)^{-1} h\,, \qquad \psi \to \omega(x)^{-3/2} \psi\,, \tag{1.3}$$

[1] See [3] for a discussion of cosmology in the context of global conformal invariance.

[2] In a sense this breaking is inevitable, like that of the diffeomorphism invariance. For finding a theory with a conformally invariant phase one should go to theories in which the metric is a derived quantity, like metric-affine gauge theories of gravity [4].

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the action of the SM is not conformally invariant. Specifically, the only symmetry violating terms are the Higgs kinetic term and the Higgs mass term. In the fermion kinetic term, the transformation of the spin connection compensates the derivative terms originating from the transformation of the fermion fields.

If we want to construct a lagrangian invariant under the conformal transformation (1.1) and still recover Einstein gravity we must introduce in the theory a compensator field, transforming under this symmetry like

$$\phi \to \omega(x)^{-1}\phi\,. \tag{1.4}$$

In this case we can write down an action for gravity [8], invariant under (1.1) and (1.4), as[3]

$$S_{\text{gravity}} = \int \mathrm{d}^4x\sqrt{g}\left\{\phi^2 R + 6\partial_\mu\phi\partial^\mu\phi\right\}. \tag{1.5}$$

We can now also use the field ϕ to construct a conformally invariant SM lagrangian as

$$\begin{aligned} S_{SM} = -\int \mathrm{d}^4x\sqrt{g}\Big\{&\phi^2\mathcal{D}_\mu h\mathcal{D}^\mu h^\dagger + \phi^4 V(h) + \frac{1}{4}F^{(a)}_{\mu\nu}F^{(a)\mu\nu} + \\ &+ \phi^{3/2} i\overline{\psi}^i\gamma^\mu(x)(\mathcal{D}_\mu + \Gamma_\mu(x))(\phi^{3/2}\psi^i) + \lambda_i\phi^4\overline{\psi}^i_L.h\psi^i_R + \text{h.c.}\Big\}, \end{aligned} \tag{1.6}$$

where the Higgs field, the gauge fields A^a_μ and the fermion fields are left invariant under the conformal transformation. One can bring the lagrangian to a more conventional form with the redefinitions $h \to h\phi^{-1}, \psi^i \to \psi^i\phi^{-3/2}$. This redefinition 'covariantizes' the derivatives in the Higgs kinetic term:

$$\phi\mathcal{D}_\mu h \to \left(\mathcal{D}_\mu - \frac{\partial_\mu\phi}{\phi}\right)h\,, \tag{1.7}$$

and reduces all the other terms to the conventional ones of the SM except for a coupling of the scalar compensator to the Higgs mass term. The Higgs field and the fermion fields now transform according to (1.3).

Notice that the kinetic term of the field ϕ is ghostlike. This does not pose any problems since the field actually is a ghost: its excitations can be gauged away by conformal transformations. Assuming ϕ is not zero, we can use our conformal gauge freedom to go to a gauge in which ϕ is constant, (applying the conformal transformation with $\omega = \phi/M_p$) and we recover in this conformal "unitary" gauge the equations of the SM coupled to Einstein gravity. So, after the assumption of a non-zero VEV for ϕ, this realization of conformal invariance appears to be devoid of physical meaning at the classical level since it just corresponds to taking the conventional action of the SM and GR, substitute in it $g_{\mu\nu} \to \phi^2 g_{\mu\nu}$ and take independent variations with respect to $g_{\mu\nu}$ and ϕ. The equation of motion of ϕ is not independent of the rest, since it is just the trace of the Einstein equations. We have introduced a new degree of freedom (and a new symmetry) only to subtract it using our conformal gauge freedom.

[3]Our conventions are $\mathrm{diag}(g_{\mu\nu}) = (-+++)$, $R^\rho{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^\rho_{\nu\mu} + \cdots$, $R = R^{\rho\mu}{}_{\rho\mu}$.

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There exists a different realization of conformal invariance in 4D that actually adds some new degrees of freedom to the gravity sector, and makes use of a vector field,[4] besides the scalar compensator,[5] to gauge the conformal invariance (see also [9] for a discussion of a gauging of conformal invariance that makes use of the transformation properties of the Ricci tensor).

So we have seen that in 4D there are ways of implementing a symmetry that conformally transforms the metric (in a realistic model) at the cost of introducing a scalar that also transforms under this symmetry and assuming a VEV for it. This expectation value is the mass scale against which we can make dimensional measurements in our theory. The assumption of a VEV for this field is crucial for obtaining a realistic model, but this might be regarded as an *ad hoc* way of realizing the conformal symmetry since it is not clear how to obtain the VEV for this field as the result of a minimization process.[6] So one might wonder: is it possible to construct a realistic theory invariant under a conformal transformation of the metric only? (In a theory in which other fields, besides the metric, have non-zero weight under conformal transformations, a positive answer to this question would mean that one can construct viable models even if these other fields are set to zero in the background.) In such a model all spacetimes related by a transformation like (1.1), *with other fields constant*, would be describing the same state so the geometry of the (full) spacetime would not be observable. Only conformally invariant quantities have physical meaning. This seems quite counter-intuitive but we will see that realistic models with this property can be built by assuming the existence of compactified extra dimensions. The size of the extra dimensions provides us with a mass scale against which we can make (relative) dimensionful measurements in the 4D effective theory.

In the next section we consider conformal theories of gravity in higher dimensions. We consider as the action for gravity the most general lagrangian built out of the metric invariant under the transformation (1.1). We show that these theories generically admit compactifications of the extra dimensions in a constant curvature manifold while the non-compact dimensions can have positive, negative or zero curvature. We provide explicit examples in six and eight dimensions. Furthermore we show that Einstein gravity is recovered at low energies, and the conformal invariance is non-linearly realized in 4D with a compensator field that has a VEV. This field corresponds to the size of the extra dimensions (the radion) and its excitations can be "gauged away" by choosing the Einstein frame for the metric. There is then no need to consider a stabilization mechanism for this modulus field in this kind of compactification.

[4]This vector field was first introduced by Weyl [5] who tried to identify it with the electromagnetic potential, and it is sometimes called the Weylon.

[5]It has been suggested that in this case, economically, the Higgs could be the field taking the role of the conformal compensator [10], in which case there would be no physical Higgs in the spectrum. All its degrees of freedom would be "eaten" in the electroweak and conformal symmetry breaking processes by the massive vector bosons.

[6]Notice however [11], where quantumgravitational corrections give a VEV to a conformally coupled scalar field in the presence of Weyl gravity, thereby inducing Einstein gravity at low energies.

2. Higher dimensional conformal gravity

Let us now consider a general conformally invariant action in an even number (D) of dimensions

$$S = \int d^D\mathrm{x}\sqrt{G}\,(\mathcal{L}_{\text{gravity}} + \delta\mathcal{L}_{\text{matter}})\,. \tag{2.1}$$

$\mathcal{L}_{\text{gravity}}$, generating the left hand side of the equations of motion (EOM), will consist of a linear combination of all conformally invariant local scalar densities that can be built out of the metric (see appendix). The number of independent conformally invariant terms increases rapidly with the dimension: in 6 dimensions there are 3 such terms [12], while in 8 dimensions there are already 12 terms [13]. (It stills remain to be seen if they are all independent at the level of the EOM, $i.e$ if we put the total derivatives to zero [14].) $\delta\mathcal{L}_{\text{matter}}$, producing the right hand side of the EOM, in the form of the energy-momentum tensor, will consist of that part of the matter lagrangian responsible for the compactification.

We will first look for compactifications into a spacetime background with a factorizable metric like

$$ds^2 = G_{MN}d\mathrm{x}^M d\mathrm{x}^N = g_{\mu\nu}(x)dx^\mu dx^\nu + \gamma_{ij}(z)dz^i dz^j \tag{2.2}$$

where $g_{\mu\nu}$ is the 4D lorentzian metric of a maximally symmetric manifold, with curvature R_g, and γ_{ij} is the metric for a compact euclidean n-dimensional maximally symmetric space ($n = D - 4$), with curvature R_γ. For this metric the gravity lagrangian will take the form (see appendix)

$$\mathcal{L}_{\text{gravity}} = \mathcal{L}(R_g, R_\gamma) = \sum_{i=0}^{D/2} c_i R_g^{D/2-i} R_\gamma^i\,. \tag{2.3}$$

One can now easily obtain the left hand side of the EOM, by considering variations $g_{\mu\nu} \to g_{\mu\nu}(1+\epsilon)$ or $\gamma_{\mu\nu} \to \gamma_{\mu\nu}(1+\epsilon)$, for constant ϵ:

$$\begin{aligned} W_{MN} &= \begin{pmatrix} g_{\mu\nu}(\frac{R_g}{4}\frac{\partial\mathcal{L}}{R_g} - \frac{\mathcal{L}}{2}) & \\ & \gamma_{ij}(\frac{R_\gamma}{n}\frac{\partial\mathcal{L}}{R_\gamma} - \frac{\mathcal{L}}{2}) \end{pmatrix} \\ &= \begin{pmatrix} g_{\mu\nu}(\sum_{i=0}^{D/2}\frac{c_i}{4}(\frac{D}{2}-i-2)R_g^{D/2-i}R_\gamma^i) & \\ & -\gamma_{ij}\frac{4}{n}(\sum_{i=0}^{D/2}\frac{c_i}{4}(\frac{D}{2}-i-2)R_g^{D/2-i}R_\gamma^i) \end{pmatrix}. \end{aligned} \tag{2.4}$$

Notice that, because of the conformal symmetry, $W_N^N = 0$. The energy-momentum tensor will also be traceless, for the same reason. We will show later that, with a proper choice for $\delta\mathcal{L}_{\text{matter}}$, one indeed produces an energy-momentum tensor of the form:

$$T_{MN} = \begin{pmatrix} -g_{\mu\nu}\Lambda & \\ & \gamma_{ij}\Lambda\frac{4}{n} \end{pmatrix}. \tag{2.5}$$

So, as a consequence of the conformal invariance, the two EOM one has in the case of conventional factorizable compactifications (of Einstein gravity), now boil down to one. For a given Λ, the curvatures R_γ and R_g are not uniquely determined. It is the (conformally invariant) ratio of Λ and $R_\gamma^{D/2}$ that determines the curvature of the 4D world. In fact it does not make sense to talk of a value for Λ until we have fixed the conformal gauge, since

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under a conformal transformation this parameter changes as $\Lambda \to \omega^{-D}\Lambda$. We can fix the conformal gauge by requiring R_γ to be an arbitrary constant, and it is then apparent that only for a particular value of Λ in this gauge, namely $\Lambda = \frac{c_{D/2}}{2} R_\gamma^{D/2}$, we recover flat 4D space.

We will now consider some examples of conformally invariant matter lagrangians in 6 or 8 dimensions that produce an energy-momentum tensor as in (2.5). Just as in Einstein gravity we will use n-forms [15] or scalars [16] to produce the spontaneous compactification, but we will have to add them in such a way that the conformal invariance is preserved. Any matter field will be considered invariant under the conformal symmetry, holding the promise of keeping the transformation (1.1) as a purely geometric space-time transformation.

In D dimensions a natural field to consider is an antisymmetric $(\frac{D}{2} - 1)$-form ($\mathbf{A}$) whose action

$$\delta S_{\text{matter}} = -\frac{1}{D/2!} \int \mathrm{d}^D \mathrm{x} \sqrt{G} H^{M..N} H_{M..N} \,, \tag{2.6}$$

with $\mathbf{H} = d\mathbf{A}$, is invariant under (1.1). So in 8D we can just consider a 3-form field A_{MNP} and consider a vacuum expectation value of its field strength as

$$\begin{aligned} H_{\mu\nu\lambda\sigma} &= \sqrt{g} E \epsilon_{\mu\nu\lambda\sigma} \\ H_{ijkl} &= \sqrt{\gamma} B \epsilon_{ijkl} \end{aligned} \tag{2.7}$$

with ϵ_{abcd} the Levi-Civita symbol with 4 indices, E and B constants and the rest of the components of $\mathbf{H}$ zero. This ansatze solves the equation of motion for the form, and yields an energy-momentum tensor given by

$$T_{MN} = \begin{pmatrix} -g_{\mu\nu} \frac{E^2+B^2}{2} & \\ & \gamma_{ij} \frac{E^2+B^2}{2} \end{pmatrix} . \tag{2.8}$$

If we now take for instance $\mathcal{L}_{\text{gravity}} = \alpha (C_{ABCD} C^{ABCD})^2$, we can use the value of this lagrangian in the background (2.2) ($\mathcal{L}_6^{(8)}$in the appendix) to obtain the EOM:

$$\frac{\alpha}{441} (R_g + R_\gamma)^3 (R_\gamma - R_g) = E^2 + B^2 \,. \tag{2.9}$$

This gives a flat 4D space if we tune the fluxes to

$$E^2 + B^2 = \frac{\alpha}{441} R_\gamma^4 \,. \tag{2.10}$$

Notice that for this value of the fluxes, one also has the real solution $R_g = R_g^\star = 0.84 R_\gamma$. It is tempting to speculate on the construction of a inflationary scenario of the type discussed in [17] that might arise from dynamical solutions starting in $R_g \approx R_g^\star$ and ending up in $R_g \approx 0$.

In the 6D case, the corresponding conformally invariant 2-form A_{MN} would naturally produce a (3+3)-splitting of the energy-momentum tensor, as opposed to the (4+2) we would like to find in order to generate the compactification to 4D. We should then consider more complicated interactions in order to yield the required energy-momentum tensor. As an example we will consider here three possibilities that involve the addition of a U(1) vector field (A_M) and a (spherical or hyperbolic) sigma model with two scalar fields (Φ^i). The conformally invariant action is[7]

$$\delta S_{\rm matter} = -\int d^6x\,\sqrt{G}\Big\{\frac{\lambda}{4}\left(f_{ij}(\Phi)\partial_M\Phi^i\partial^M\Phi^j\right)^3 + \frac{\kappa}{2}f_{ij}(\Phi)\partial_M\Phi^i\partial^M\Phi^j F_{NP}F^{NP} + \\ + \beta\, C^{MNPQ}F_{MN}F_{PQ}\Big\}\,, \qquad (2.11)$$

with $F_{MN} = \partial_{[M}A_{N]}$ and f_{ij} is the metric of a manifold, with constant curvature $s = \pm 1$, parameterized by the scalar fields. For $\gamma_{ij}(x)$ proportional to $f_{ij}(x)$, it can be checked that a solution for the scalar fields and the U(1) field is simply

$$\Phi^i = z^i\,; \qquad F_{ij} = \sqrt{\gamma}B\epsilon_{ij}\,. \qquad (2.12)$$

These VEVs produce a traceless energy-momentum tensor like (2.5) with $n = 2$ where

$$\Lambda = \lambda\,|R_\gamma|^3 + \kappa B^2|R_\gamma| + \beta B^2\left(\frac{R_g}{20} + \frac{3R_\gamma}{5}\right). \qquad (2.13)$$

If we now take for instance $\mathcal{L}_{\rm gravity} = \alpha C^{A\ \ C}_{\ \ B\ \ D}C^{B\ \ D}_{\ \ E\ \ F}C^{E\ \ F}_{\ \ A\ \ C}$, the EOM reads

$$\frac{17\alpha}{57600}(R_g + 6R_\gamma)^2\left(3R_\gamma - \frac{R_g}{4}\right) = \Lambda\,. \qquad (2.14)$$

We can recover a flat 4D space by tuning the flux to:

$$B^2 = \frac{\frac{51\alpha}{1600} - s\lambda}{\frac{3}{5}\beta + s\kappa}R_\gamma^2\,. \qquad (2.15)$$

For certain values of the couplings, this flux will again produce additional real solutions with non-zero R_g.

In the apppendix we give the values of the fluxes, that induce a flat 4D space, for the most general conformally invariant action in 6 and 8 dimensions. The inclusion of the other curvature invariants does not change the qualitative aspects of the solutions. In particular including the invariants that do not have derivatives amounts only to a redefinition of α in eqs. (2.9) and (2.14) since all these invariants are proportional to each other in the backgrounds we are considering (and they are also proportional to each other for spherically symmetric backgrounds [18]).

It is also easy to see that at low energies gravity will be described by Einstein gravity with conformal invariance non-linearly realized in the action like in eq. (1.5), and the radion field will play the role of the compensator field. For seeing this we can consider the metric ansatze

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \phi(x)^{-2}\gamma_{ij}(z)dz^i dz^j \qquad (2.16)$$

with arbitrary $g_{\mu\nu}(x)$ and $\phi(x)$ (while γ_{ij} would still correspond to a compact constant curvature manifold, with curvature $s = \pm 1$) and compute the effective 4D action $S_{(4D)}(g_{\mu\nu}, \phi)$.

[7]Notice that conventional kinetic terms for Φ^i or A_M are forbidden by the conformal invariance.

This action will be invariant under the transformations $g_{\mu\nu} \to \omega(x)^2 g_{\mu\nu}$ and $\phi \to \omega(x)^{-1}\phi$ as a consequence of the higher dimensional conformal invariance, so in order to find $S_{(4D)}(g_{\mu\nu}, \phi)$ we can first compute $S_{(4D)}(g_{\mu\nu}, \phi_0)$, with ϕ_0 a constant and substitute $g_{\mu\nu} \to (\phi/\phi_0)^2 g_{\mu\nu}$ to recover $S_{(4D)}(g_{\mu\nu}, \phi)$.

However, this exercise would not be very useful, since the conformal invariance allows us to take $\phi = \phi_0$ constant without loss of generality. This corresponds to taking the Einstein frame in the 4D action, and it is clear now why in this frame the excitations of the volume of the extra dimensional manifold are the degree of freedom of the higher dimensional metric sacrificed to fix the conformal gauge. If we tune the fluxes and/or couplings of $\delta\mathcal{L}_{\rm matter}$ such that the constant term ($\propto \phi_0^4$) in $S_{(4D)}$ is cancelled, we recover Einstein gravity with a zero cosmological constant at low energies, since higher order curvature corrections have a negligible impact on low energy physics in flat space. Also, we can mention here that in the effective action, the 4D energy-momentum tensor is not traceless anymore. Conformal invariance is spontaneously broken and the tracelessness of the higher dimensional energy-momentum tensor is reflected in 4D as the fact that the equation of motion of the radion is the trace of the Einstein equations.

As an example we take the 8D action that we considered before

$$S = \int \mathrm{d}^8x\sqrt{G}\left\{\alpha\left(C_{ABCD}C^{ABCD}\right)^2 - \frac{1}{4!}H_{ABCD}H^{ABCD}\right\}. \tag{2.17}$$

In the flux background (2.7), with a spherical compactification, this yields an effective 4D action[8]

$$S_{(4D)} = V_\gamma \int \mathrm{d}^4x\sqrt{g}\left\{\alpha\left(\frac{\phi_0^4}{21} + \frac{2}{21}\phi_0^2 R + \frac{R^2}{21} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{2}{3}R_{\mu\nu}R^{\mu\nu}\right)^2 - (E^2 + B^2)\right\}. \tag{2.18}$$

If we now plug in the volume of the 4-dimensional sphere $V_\gamma = \frac{8\pi^2}{3}(\frac{12}{\phi_0^2})^2$ and tune the fluxes to the value obtained in (2.10), we indeed recover Einstein gravity:

$$S_{(4D)} = \int \mathrm{d}^4x\sqrt{g}\left(\frac{M_p^2}{16\pi}R + \cdots\right), \tag{2.19}$$

with zero cosmological constant and a Planck mass $M_p^2 = \frac{8192}{147}\alpha\pi^3\phi_0^2 = \frac{65536}{49}\alpha\pi^4\sqrt{2/(3V_\gamma)}$, set by the volume of the extra dimensional manifold.

For the 6D action that we considered before, the spherical compactification in the background (2.12), with the fine tuning (2.15), leads in a similar way to Einstein gravity with zero cosmological constant and

$$M_p^2 = 128\pi^2\phi_0^2 \frac{\alpha\frac{51}{1600}(2\beta+5\kappa)+\lambda\beta}{3\beta+5\kappa} = \frac{2(8\pi)^3}{V_\gamma}\frac{\alpha\frac{51}{1600}(2\beta+5\kappa)+\lambda\beta}{3\beta+5\kappa}. \tag{2.20}$$

[8] This effective action is not exactly the one obtained from the higher dimensional action with the ansatze substituted in it. There is a sign flip in the term with the 'electric' four-form flux, see also the comments in [19].

We see that in these models, although one would naturally expect the size and curvature of the extra dimensions to be of the order of the Planck mass, one can also have low compactification scales (if $3\beta \simeq -5\kappa$ in the previous example for instance).

The expressions of the Planck mass for the most general conformally invariant action in 6 and 8 dimensions are given in the appendix.

3. Conclusions

In this letter we have considered conformally invariant theories of gravity as possible extensions of General Relativity motivated by their higher degree of symmetry. The usual mechanism to recover Einstein gravity as a long distance effective theory, involved the introduction of a scalar that also transformed under the conformal symmetry, only to gauge it away by using our conformal gauge freedom. This seemed to spoil the geometric nature of the conformal invariance. Furthermore, one had to assume a nonzero VEV for this field, while a zero VEV is always allowed by the classical EOM (see however [11]).

We provide an alternative mechanism, that starts from a *pure* conformally invariant theory in higher dimensions. We have seen that by including a matter lagrangian that respects the conformal symmetry one can indeed find solutions in which only four dimensions remain non-compact and the rest are compactified in a highly curved manifold. By computing the lower dimensional effective action we have also shown that Einstein gravity is now recovered in 4D as a result of this so-called spontaneous compactification. This gives a geometric origin to the scalar compensator field: it is the radion, or the field associated with the size of the extra dimensions. Once we take the extra compact dimensions for granted, the nonzero VEV of this field becomes evident. If the extra dimensions have spherical topology for instance, the curvature of the extra dimensions can be arbitrarily small but never zero, so zero values for the radion are excluded. One can now safely use the conformal symmetry to fix the value of the compensator field to a constant. From the higher dimensional point of view, this reveals the impossibility of destabilization of the size of the extra dimensions, since we can always use our conformal invariance to go to a gauge in which the extra dimensions have a fixed size and curvature. It is precisely in this gauge (the so-called Einstein gauge) that we recover the canonically normalized Einstein-Hilbert lagrangian as the effective action for the metric in 4D, with the Planck mass proportional to the curvature of the extra-dimensional manifold. But although the curvature (or the size) of the extra dimensions can be considered to take a fixed value without loss of generality, there could be other sources of instability in these compactifications. The requirement of stability could restrict the available parameter space in these models. A deeper study of these issues is however beyond the scope of the present paper.

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A. Formulae for general lagrangians in 6D and 8D

In six dimensions the three conformally invariant local scalar densities that can be built out of the metric are [12]:

$$\begin{aligned}
\mathcal{L}_1^{(6)} &= C^{A\ \ C}_{\ \ B\ \ D} C^{B\ \ D}_{\ \ E\ \ F} C^{E\ \ F}_{\ \ A\ \ C} \,, \\
\mathcal{L}_2^{(6)} &= C^{A\ \ C}_{\ \ B\ \ D} C^{B\ \ E}_{\ \ A\ \ F} C^{D\ \ F}_{\ \ C\ \ E} \,, \\
\mathcal{L}_3^{(6)} &= C^{A\ \ C}_{\ \ B\ \ D} \Box C^{B\ \ D}_{\ \ A\ \ C} + 2C^{ABCD} C_{ABCE} R^E_D - 3C_{ABCD} R^{AC} R^{BD} - \\
&\quad - \frac{3}{2} R^{AB} R_{BC} R^C_A + \frac{27}{20} R^{AB} R_{AB} R - \frac{21}{100} R^3 \,.
\end{aligned} \tag{A.1}$$

In the spacetime background (2.2) they reduce to:

$$\begin{aligned}
\frac{57600}{17}\mathcal{L}_1^{(6)} &= -\frac{14400}{13}\mathcal{L}_2^{(6)} = (R_g + 6R_\gamma)^3 \,, \\
\mathcal{L}_3^{(6)} &= \frac{41}{2400} R_g^3 - \frac{117}{400} R_g^2 R_\gamma + \frac{129}{200} R_g R_\gamma^2 + \frac{9}{100} R_\gamma^3 \,.
\end{aligned} \tag{A.2}$$

For the most general action:

$$\int d^6x \sqrt{G} \sum_{i=1}^{3} \alpha_i \mathcal{L}_i^{(6)} \,, \tag{A.3}$$

$\delta S_{\mathrm{matter}}$ (2.11) induces a compactification to flat 4D-space if the flux is tuned to:

$$B^2 = \frac{108\tilde{\alpha} + \frac{9}{200}\alpha_3 - s\lambda}{\frac{3}{5}\beta + s\kappa} R_\gamma^2 \,, \tag{A.4}$$

with $\tilde{\alpha} \equiv \frac{17}{57600}\alpha_1 - \frac{13}{14400}\alpha_2$. In the case of a spherical compactification, this results in low energy Einstein gravity with zero cosmological constant and a Planck mass:

$$\begin{aligned}
M_p^2 &= 128\pi^2 \phi_0^2 \frac{108\tilde{\alpha}(2\beta + 5\kappa) + \alpha_3 \frac{378\beta + 645\kappa}{200} + \lambda\beta}{3\beta + 5\kappa} \\
&= \frac{2(8\pi)^3}{V_\gamma} \frac{108\tilde{\alpha}(2\beta + 5\kappa) + \alpha_3 \frac{378\beta + 645\kappa}{200} + \lambda\beta}{3\beta + 5\kappa} \,.
\end{aligned} \tag{A.5}$$

The twelve independent Weyl invariants in eight dimensions were obtained in [13]. We refer to this paper for the expressions of the lagrangians in terms of Weyl tensors and Weyl covariant derivatives [20] and use the same numbering. Five invariants ($\mathcal{L}_1^{(8)} - \mathcal{L}_5^{(8)}$) contain derivatives, the others ($\mathcal{L}_6^{(8)} - \mathcal{L}_{12}^{(8)}$) only contain contractions of Weyl tensors. In the background (2.2), $\mathcal{L}_1^{(8)}$ reduces to:

$$\mathcal{L}_1^{(8)} = (R_g + R_\gamma)^2 \frac{254R_g^2 + 1243R_g R_\gamma + 254R_\gamma^2}{123480} \,. \tag{A.6}$$

All the other invariants are proportional to the same term:

$$\mathcal{L}_i^{(8)} = c_i (R_g + R_\gamma)^4 \,, \tag{A.7}$$

with

$$c_2 = -\frac{5}{12348} \,; \quad c_3 = \frac{25}{3087} \,; \quad c_4 = \frac{5}{56448} \,; \quad c_5 = -\frac{1}{98784} \,; \quad c_6 = \frac{1}{441} \,; \quad c_7 = \frac{1}{3528}$$

$$c_8 = \frac{13}{148176}\,; \quad c_9 = \frac{13}{296352}\,; \quad c_{10} = \frac{13}{592704}\,; \quad c_{11} = \frac{509}{1185408}\,; \quad c_{12} = \frac{25}{296352}\,. \tag{A.8}$$

For the most general conformally invariant action,

$$\int d^8x\sqrt{G}\sum_{i=1}^{12}\alpha_i\mathcal{L}_i^{(8)}\,, \tag{A.9}$$

the 3-form field (2.6) will now induce a compactification to flat 4D space if we tune the fluxes to:

$$E^2 + B^2 = \left(\frac{127}{61740}\alpha_1 + \tilde{\alpha}\right) R_\gamma^4\,, \tag{A.10}$$

with $\tilde{\alpha} \equiv \sum_{i=2}^{12} c_i\alpha_i$. A spherical compactification then results in low energy Einstein gravity with Planck mass:

$$\begin{aligned} M_p^2 &= 6144\pi^3\phi_0^2\left(\frac{1751}{123480}\alpha_1 + 4\tilde{\alpha}\right)\,, \\ &= 147456\pi^4\sqrt{\frac{2}{3V_\gamma}}\left(\frac{1751}{123480}\alpha_1 + 4\tilde{\alpha}\right)\,. \end{aligned} \tag{A.11}$$

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